# ON THE GLOBAL WELL-POSEDNESS FOR THE AXISYMMETRIC EULER EQUATIONS

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ABSTRACT. This paper deals with the global well-posedness of the 3D axisymmetric Euler equations for initial data lying in critical Besov spaces $B^{1+3/p}_{p,1}$. In this case the BKM criterion is not known to be valid and to circumvent this difficulty we use a new decomposition of the vorticity.

## 1. Introduction

The evolution of homogeneous inviscid incompressible fluid flows in $\mathbb{R}^3$ is governed by the Euler system

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Here, $u = u(t,x) \in \mathbb{R}^3$ denotes the velocity of the fluid, the scalar function $\pi = \pi(t,x)$ stands for the scalar pressure and $u \cdot \nabla = \sum_{j=1}^3 u^j \partial_j$.
The local theory of the system (1.1) seems to be in a satisfactory state and several results are obtained by numerous authors in many standard function spaces. In [9], Kato proved the local existence and uniqueness for initial data $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$ and Chemin [5] gave similar results for initial data lying in Hölderian spaces $C^r$ with $r > 1$.
Other local results are recently obtained by Chae [4] in critical Besov spaces $B^{1+3/p}_{p,1}$, with $p \in ]1,\infty[$ and by Pak and Park [11] for the space $B^1_{\infty,1}$. Notice that these spaces have the same scaling as Lipschitz functions (the space which is relevant for the hyperbolic theory) and in this sens they are called critical.
The question of global existence (even for a smooth initia data) is still open and continues to be one of the most leading problem in mathematical fluid mechanics. The well-known BKM criterion [1] ensures that the development of finite time singularities for Kato's solutions is related to the blowup of the $L^\infty$ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of two-dimensional Euler solutions for smooth initial data since the vorticity is only advected

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and then does not grow. We emphasize that new geometric blowup criteria are recently discovered by Constantin, Fefferman and Majda [6].
Let us recall that, in space dimension three, the vorticity is defined by the vector $\omega = \operatorname{curl} u$ and satisfies the equation

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0.$$

The main difficulty for establishing global regularity is to understand how the vortex stretching term $(\omega\cdot\nabla)u$ affects the dynamic of the fluid.

While global existence is not proved for arbitrary initial smooth data, there are partial results in the case of the so-called axisymmetric flows without swirl. We say that a vector field $u$ is axisymmetric if it has the form:

$$u(x,t) = u^r(r,z,t)e_r + u^z(r,z,t)e_z, \quad x=(x_1,x_2,z), \quad r=(x_1^2+x_2^2)^{\frac{1}{2}},$$

where $\big(e_r, e_\theta, e_z\big)$ is the cylindrical basis of $\mathbb{R}^3$ and the components $u^r$ and $u^z$ do not depend on the angular variable. The main feature of axisymmetric flows arises in the vorticity which takes the form (more precise discussion will be done in Proposition 3.1 and 3.2),

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta$$

and satisfies

$$\partial_t\omega + (u\cdot\nabla)\omega = \frac{u^r}{r}\omega. \tag{1.2}$$

Consequently the quantity $\alpha := \omega/r$ is only advected by the flow, that is

$$\partial_t\alpha + (u\cdot\nabla)\alpha = 0. \tag{1.3}$$

This fact induces the conservation of all the norms $\|\alpha\|_{L^p}, 1 \leq p \leq \infty$. In [16], Ukhovskii and Yudovich took advantage of these conservation laws to prove the global existence for axisymmetric initial data with finite energy and satisfying in addition $\omega_0 \in L^2\cap L^\infty$ and $\frac{\omega_0}{r} \in L^2\cap L^\infty$. In terms of Sobolev regularity these assumptions are satisfied if the velocity $u_0$ belongs to $H^s$ with $s > \frac{7}{2}$. This is far from the critical regularity of local existence theory $s = \frac{5}{2}$. The optimal result in Sobolev spaces is done by Shirota and Yanagisawa in [15] who proved global existence in $H^s$, with $s > \frac{5}{2}$. Their proof is based on the boundness of the quantity $\|\frac{u^r}{r}\|_{L^\infty}$ by using Biot-Savart law. We mention also the reference [13] where similar results are given in different function spaces. In a recent work [7], Danchin has weakened the Ukhoviskii and Yudovich conditions. More precisely, he obtains global existence and uniqueness for initial data $\omega_0 \in L^{3,1}\cap L^\infty$ and $\frac{\omega_0}{r} \in L^{3,1}$. Here, we denote by $L^{3,1}$ the Lorentz space.

In this paper we address the question of global existence in the critical spaces $B^{1+3/p}_{p,1}$. Comparing to the sub-critical spaces this problem is extremely hard to deal with because we are deprived of an important tool which is the BKM criterion. Even in space dimension two we encounter the

same problem. Although the quantity $\|\omega(t)\|_{L^\infty}$ is conserved, this is not sufficient to propagate for all time the initial regularity. As it was pointed by Vishik in [17] the significant quantity is $\|\omega(t)\|_{B^0_{\infty,1}}$ and its control needs the use of the special structure of the vorticity, which is only transported by the flow.

Owing to the streching term $\omega\, u^r/r$, the estimate of $\|\omega(t)\|_{B^0_{\infty,1}}$ for axisymmetric flows is more complicated and needs as we shall see a refined analysis of the geometric structure of the vorticity.
The main result of this paper can be stated as follows (for the definition of function spaces see next section).

**Theorem 1.1.** *Assume $p \in [1,\infty]$. Let $u_0$ be an axisymmetric divergence free vector field belonging to $B^{1+3/p}_{p,1}$, such that its vorticity satisfies $\omega_0/r \in L^{3,1}$. Then the system* (1.1) *has a unique global solution $u \in \mathcal{C}(\mathbb{R}_+;\, B^{1+3/p}_{p,1})$.*

*Remark* 1.2. We mention that for $p<3$ the condition $\frac{\omega_0}{r} \in L^{3,1}$ is automatically derived from $u_0 \in B^{1+3/p}_{p,1}$ (see Proposition 2.2 below).

*Remark* 1.3. In the proof of this theorem we have established the following global in time estimates

$$\|u(t)\|_{B^{s_p}_{p,1}} \leq C_0 e^{e^{\exp C_0 t}} \quad \text{and} \quad \|\omega(t)/r\|_{L^{3,1}} \leq \|\omega_0/r\|_{L^{3,1}},$$

where the constant $C_0$ depends only on the initial data norms.

The proof is heavily related to two crucial estimates, the first one is the $L^\infty$ bound of the vorticity for every time which is obtained from Biot-Savart law and the use of Lorentz spaces, see Proposition 4.1. Unfortunately as it has been discussed above this is not sufficient to show global existence because we do not know whether the BKM criterion works in the critical spaces or not. Thus we are led to establish a second new estimate for the vorticity in Besov space $B^0_{\infty,1}$ (see Proposition 4.4).This allows us to bound for every time the Lipschitz norm of the velocity which is sufficient to prove global existence.The control of $\|\omega(t)\|_{B^0_{\infty,1}}$ is the most important part of this paper and it is done in a non fashion way in which the axisymmetric geometry plays a key role.

The rest of this paper is organized as follows. In section 2 we recall some function spaces and gather some preliminary estimates. Section 3 is devoted to the study of some geometric properties of any solution to a vorticity equation model. The proof of Theorem 1.1 is done in several steps in section 4.

## 2. Notations and preliminaries

Throughout this paper, $C$ stands for some real positive constant which may be different in each occurrence. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$.

• Let us start with a classical dyadic decomposition of the full space (see for instance [5]): there exist two radial functions $\chi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3$,

ii) $\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{if} \quad \xi \neq 0$,

iii) $\text{supp } \varphi(2^{-p}\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \varnothing$, if $|p-q| \geq 2$,

iv) $q \geq 1 \Rightarrow \text{supp}\chi \cap \text{supp } \varphi(2^{-q}) = \varnothing$.

For every $u \in \mathcal{S}'(\mathbb{R}^3)$ one defines the nonhomogeneous Littlewood-Paley operators by,

$$\Delta_{-1}u = \chi(\mathrm{D})u;\ \forall q \in \mathbb{N},\ \Delta_q u = \varphi(2^{-q}\mathrm{D})u \quad \text{and} \quad S_q u = \sum_{-1\leq j\leq q-1} \Delta_j u.$$

One can easily prove that for every tempered distribution $u$,

$$u = \sum_{q\geq -1} \Delta_q\, u. \tag{2.1}$$

The homogeneous operators are defined as follows

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q u = \sum_{j\leq q-1} \dot{\Delta}_j u.$$

We notice that these operators can be written as a convolution. For example for $q \in \mathbb{Z}$, $\dot{\Delta}_q u = 2^{3q} h(2^q\cdot) \star u$, where $h \in \mathcal{S}$ and $\widehat{h}(\xi) = \varphi(\xi)$.
For the homogeneous decomposition, the identity (2.1) is not true due to the polynomials but we have,

$$u = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the whole of polynomials (see [12]).
• In the sequel we will make an extensive use of Bernstein inequalities (see for example [5]).

**Lemma 2.1.** *There exists a constant $C$ such that for $k \in \mathbb{N}$, $1 \leq a \leq b$ and $\psi \in L^a$, we have*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q \psi\|_{L^b} \quad \leq \quad C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))} \|S_q\psi\|_{L^a},$$

*and*

$$C^{-k}2^{qk}\|\dot\Delta_q\psi\|_{L^a} \quad\leq\quad \sup_{|\alpha|=k}\|\partial^\alpha\dot\Delta_q\psi\|_{L^a}\leq C^k 2^{qk}\|\dot\Delta_q\psi\|_{L^a}.$$

Let us now introduce the basic tool of the paradifferential calculus which is Bony's decomposition [3]. It distinguishes in a product $uv$ three parts as follows:

$$uv = T_u v + T_v u + \mathcal{R}(u,v),$$

where

$$T_u v = \sum_q S_{q-1}u\Delta_q v, \quad \text{and} \quad \mathcal{R}(u,v) = \sum_q \Delta_q u\widetilde\Delta_q v,$$

$$\text{with} \quad \widetilde\Delta_q = \sum_{i=-1}^{1}\Delta_{q+i}.$$

$T_u v$ is called paraproduct of $v$ by $u$ and $\mathcal{R}(u,v)$ the remainder term.
Let $(p,r)\in[1,+\infty]^2$ and $s\in\mathbb{R}$, then the nonhomogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions $u$ such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

We remark that we have the identification $B^s_{2,2} = H^s$. Also, by using the Bernstein inequalities we get easily

$$B^s_{p_1,r_1}\hookrightarrow B^{s+3(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}, \qquad p_1\leq p_2 \quad \textit{and} \quad r_1\leq r_2.$$

• For a measurable function $f$ we define its nonincreasing rearrangement by

$$f^*(t) := \inf\Big\{s,\ \mu\big(\{x,\ |f(x)|>s\}\big)\leq t\Big\},$$

where $\mu$ denotes the usual Lebesgue measure. For $(p,q)\in[1,+\infty]^2$, the Lorentz space $L^{p,q}$ is the set of functions $f$ such that $\|f\|_{L^{p,q}}<\infty$, with

$$\|f\|_{L^{p,q}} := \begin{cases} \Big(\displaystyle\int_0^\infty [t^{\frac{1}{p}}f^*(t)]^q\frac{dt}{t}\Big)^{\frac{1}{q}}, & \text{for } 1\leq q<\infty \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}}f^*(t), & \text{for } q=\infty. \end{cases}$$

Notice that we can also define Lorentz spaces by real interpolation from Lebesgue spaces:

$$(L^{p_0},L^{p_1})_{(\theta,q)} = L^{p,q},$$

where $1\leq p_0<p<p_1\leq\infty$, $\theta$ satisfies $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $1\leq q\leq\infty$. We have the classical properties:

$$\|uv\|_{L^{p,q}}\leq\|u\|_{L^\infty}\|v\|_{L^{p,q}}. \tag{2.2}$$

$$L^{p,q}\hookrightarrow L^{p,q'}, \forall\, 1\leq p\leq\infty; 1\leq q\leq q'\leq\infty \quad \text{and} \quad L^{p,p}=L^p. \tag{2.3}$$

The next proposition precises the statement of Remark 1.2.

**Proposition 2.2.** *Let $1 \leq p < 3$ and $u \in B^{1+3/p}_{p,1}(\mathbb{R}^3)$ be an axisymmetric divergence free vector field. If we denote by $\omega$ its vorticity, then we have*

$$\|\omega/r\|_{L^{3,1}} \lesssim \|u\|_{B^{1+3/p}_{p,1}}.$$

*Proof.* First we start with showing the embedding $B^{\frac{3}{p}-1}_{p,1} \hookrightarrow L^{3,1}$. Let $(p,r)$ be a fixed exponents pair satisfying $1 \leq p < 3 < r \leq \infty$. By definition, we have

$$(L^p, L^r)_{(\theta,1)} = L^{3,1}, \quad \text{with} \quad \frac{1}{3} = \frac{1-\theta}{p} + \frac{\theta}{r}\cdot$$

According to Bernstein inequalities, we have

$$B^{\frac{3}{p}-\frac{3}{r}}_{p,1} \hookrightarrow B^0_{r,1} \hookrightarrow L^r \quad \text{and} \quad B^0_{p,1} \hookrightarrow L^p.$$

Consequently,

$$(B^0_{p,1}, B^{\frac{3}{p}-\frac{3}{r}}_{p,1})_{(\theta,1)} \hookrightarrow L^{3,1}.$$

On the other hand, we have (see for instance [2] page 152),

$$(B^0_{p,1}, B^{\frac{3}{p}-\frac{3}{r}}_{p,1})_{(\theta,1)} = B^{\theta(\frac{3}{p}-\frac{3}{r})}_{p,1} = B^{\frac{3}{p}-1}_{p,1}.$$

This completes the proof of the embedding $B^{\frac{3}{p}-1}_{p,1} \hookrightarrow L^{3,1}$. From this it ensures

$$\begin{aligned}\|\nabla\omega\|_{L^{3,1}} &\lesssim \|\nabla\omega\|_{B^{\frac{3}{p}-1}_{p,1}}\\ &\lesssim \|u\|_{B^{1+3/p}_{p,1}}.\end{aligned}$$

It remains to show the following estimate

$$\|\omega/r\|_{L^{3,1}} \lesssim \|\nabla\omega\|_{L^{3,1}}.$$

Since $B^{\frac{3}{p}}_{p,1} \hookrightarrow B^0_{\infty,1} \hookrightarrow C^0$, then $\omega$ is a continuous function and Proposition 3.1 implies $\omega(0,0,z) = 0$. By a standard smoothing procedure we may assume that $\omega$ is sufficiently smooth. According to Taylor formula we write

$$\omega(x_1,x_2,z) = \int_0^1 \Big(x_1\partial_{x_1}\omega(\tau x_1,\tau x_2,z) + x_2\partial_{x_2}\omega(\tau x_1,\tau x_2,z)\Big)d\tau.$$

Therefore we obtain from (2.2) and by homogeneity

$$\begin{aligned}\|\omega/r\|_{L^{3,1}} &\lesssim \int_0^1 \|\nabla\omega(\tau\cdot,\tau\cdot,\cdot)\|_{L^{3,1}}d\tau\\ &\lesssim \|\nabla\omega\|_{L^{3,1}}\int_0^1 \tau^{-\frac{2}{3}}d\tau\\ &\lesssim \|\nabla\omega\|_{L^{3,1}}.\end{aligned}$$

This achieves the proof. □

The following result will be needed.

**Proposition 2.3.** *Given $(p,q)\in[1,\infty]^2$ and a smooth divergence free vector field $u$. Let $f$ be a smooth solution of the transport equation*

$$\partial_t f + u\cdot\nabla f = 0,\ f_{|t=0} = f_0.$$

*Then we have*

$$\|f(t)\|_{L^{p,q}} \leq \|f_0\|_{L^{p,q}}.$$

*Proof.* We use the conservation of the Lebesgue norms combined with a standard interpolation argument, see for instance [2]. □

## 3. Geometric properties of the vorticity

In this section we will describe some special geometric properties of axisymmetric flows. The following is classical and for the convenience of the reader we give the proof.

**Proposition 3.1.** *Let $u=(u^1,u^2,u^3)$ be a smooth axisymmetric vector field. Then we have*

i) *the vector $\omega=\nabla\times u=(\omega^1,\omega^2,\omega^3)$ satsifies $\omega\times e_\theta=(0,0,0)$. In particular, we have for every $(x_1,x_2,z)\in\mathbb{R}^3$,*

$$\omega^3=0,\quad x_1\omega^1(x_1,x_2,z)+x_2\omega^2(x_1,x_2,z)=0\quad \textit{and}$$

$$\omega^1(x_1,0,z)=\omega^2(0,x_2,z)=0.$$

ii) *for every $q\geq -1$, $\Delta_q u$ is axisymmetric and*

$$(\Delta_q u^1)(0,x_2,z)=(\Delta_q u^2)(x_1,0,z)=0.$$

*Proof.* i) The first point is easy to show since the vorticity is given by

$$\omega=\begin{pmatrix}\sin\theta\partial_r u^3-\sin\theta\partial_3 u^r\\ \cos\theta\partial_3 u^r-\cos\theta\partial_r u^3\\ 0\end{pmatrix}=(\partial_3 u^r-\partial_r u^3)e_\theta.$$

The other properties are a direct consequence from this information.

ii) To prove that the angular component $(\Delta_q u)^\theta$ is zero it suffices to show that $x_2\Delta_q u^1 - x_1\Delta_q u^2=0$. In Fourier variables it is equivalent to the identity $\partial_{\xi_2}(\widehat{\Delta_q u^1})-\partial_{\xi_1}(\widehat{\Delta_q u^2})=0$. Since $\varphi$ is radial then we get

$$\begin{aligned}\partial_{\xi_2}(\widehat{\Delta_q u^1})-\partial_{\xi_1}(\widehat{\Delta_q u^2}) &= \varphi(2^{-q}|\xi|)(\partial_{\xi_2}\widehat{u^1}-\partial_{\xi_1}\widehat{u^2})\\ &+\ 2^{-q}|\xi|^{-1}\varphi'(2^{-q}|\xi|)\big(\xi_2\widehat{u^1}-\xi_1\widehat{u^2}\big).\end{aligned}$$

Since $u^\theta=0$ then $x_1u^2-x_2u^1=0$, and consequently the first term of the right hand side is zero. Thus we find

$$\begin{aligned}\partial_{\xi_2}(\widehat{\Delta_q u^1})-\partial_{\xi_1}(\widehat{\Delta_q u^2}) &= 2^{-q}|\xi|^{-1}\varphi'(2^{-q}|\xi|)\big(\xi_2\widehat{u^1}-\xi_1\widehat{u^2}\big)\\ &= -i2^{-q}|\xi|^{-1}\varphi'(2^{-q}|\xi|)\widehat{\omega^3}\\ &= 0.\end{aligned}$$

It follows that $\Delta_q u = (\Delta_q u)^r e_r + \Delta_q u^3 e_z$. To end the proof it remains to show that both components do not depend on the angle $\theta$. For this purpose it suffices to have

$$(\Delta_q u)^r(\mathcal{R}_\eta x) = (\Delta_q u)^r(x) \quad \text{and} \quad \Delta_q u^3(\mathcal{R}_\eta x) = \Delta_q u^3(x),$$

where $\mathcal{R}_\eta$ is the rotation with angle $\eta$ and axis $(Oz)$. It is easy to see from the definition that

$$(\Delta_q u)^r(x) = 2^{3q}\int_{\mathbb{R}^3} h(2^q(x-y))u^r(y)\, e_r(y)\cdot e_r(x)dy.$$

Since $h$ is radial and the rotation preserves angles and distances then we get

$$\begin{aligned}(\Delta_q u)^r(\mathcal{R}_\eta x) &= 2^{3q}\int_{\mathbb{R}^3} h(2^q\mathcal{R}_\eta(x-y))u^r(\mathcal{R}_\eta y)\, e_r(\mathcal{R}_\eta y)\cdot e_r(\mathcal{R}_\eta x)dy\\ &= 2^{3q}\int_{\mathbb{R}^3} h(2^q(x-y))u^r(\mathcal{R}_\eta y)\, e_r(y)\cdot e_r(x)dy\\ &= (\Delta_q u)^r(x).\end{aligned}$$

We have used in the last line the fact that $u^r(\mathcal{R}_\eta y) = u^r(y)$ which is an easy consequence of the axisymmetry of the flow. By the same way we obtain $\Delta_q u^3(\mathcal{R}_\eta x) = \Delta_q u^3(x)$. This achieves the proof. $\square$

The last part of this section is dedicated to the study a vorticity equation type in which no relations between the vector field $u$ and the solution $\Omega$ are supposed. More precisely, we consider

$$\left\{\begin{array}{l}\partial_t\Omega + (u\cdot\nabla)\Omega = \Omega\cdot\nabla u,\\ \operatorname{div} u = 0,\\ \Omega_{|t=0} = \Omega_0.\end{array}\right. \tag{3.1}$$

We will assume that $u$ is axisymmetric and the unknown function $\Omega = (\Omega^1, \Omega^2, \Omega^3)$ is a vector field. The following result describes the preservation of some initial geometric conditions of the solution $\Omega$.

**Proposition 3.2.** *Let $u$ be a divergence free and axisymmetric vector field belonging to $L^1_{\text{loc}}(\mathbb{R}_+, \operatorname{Lip}(\mathbb{R}^3))$ and $\Omega$ the unique global solution of* (3.1) *with smooth initial data $\Omega_0$. Then the following properties hold.*

i) *If $\operatorname{div}\Omega_0 = 0$ then $\operatorname{div}\Omega(t) = 0$, for every $t\in\mathbb{R}_+$.*
ii) *If $\Omega_0\times e_\theta = (0,0,0)$ then we have*

$$\Omega(t)\times e_\theta = (0,0,0), \qquad \forall\, t\in\mathbb{R}_+.$$

*Consequently, $\Omega^1(t,x_1,0,z) = \Omega^2(t,0,x_2,z) = 0$, and*

$$\partial_t\Omega + (u\cdot\nabla)\Omega = \frac{u^r}{r}\Omega.$$

*Proof.* First, we notice that the existence and uniqueness of global solution can be done in classical way. Indeed, let $\psi$ denote the flow of the velocity $u$, that is the vector-valued function satisfying

$$\psi(t,x)=x+\int_0^t u(\tau,\psi(\tau,x))d\tau.$$

Since $u\in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathrm{Lip}(\mathbb{R}^3))$ then it follows from the ODE theory that the function $\psi$ is uniquely and globally defined. Let $\widetilde{\Omega}(t,x):=\Omega(t,\psi(t,x))$ and $A(t,x)$ the matrix such that $A(t,\psi^{-1}(t,x))=(\partial_j u_i)_{1\leq i,j\leq 3}$, then it is obvious that

$$\partial_t\widetilde{\Omega}=A(t,x)\widetilde{\Omega}.$$

From Cauchy-Lipschitz theorem this last equation has a unique global solution, and the system (3.1) too.

i) We apply the divergence operator to the equation (3.1) leading under the assumption $\operatorname{div} u=0$, to

$$\partial_t\operatorname{div}\Omega+u\cdot\nabla\operatorname{div}\Omega=0.$$

Thus, the quantity $\operatorname{div}\Omega$ is transported by the flow and consequently the incompressibility of $\Omega$ remains true for every time.

ii) We denote by $(\Omega^r,\Omega^\theta,\Omega^z)$ the coordinates of $\Omega$ in cylindrical basis. It is obvious that $\Omega^r=\Omega\cdot e_r$. Recall that in cylindrical coordinates the operator $u\cdot\nabla$ has the form

$$u\cdot\nabla=u^r\partial_r+\frac{1}{r}u^\theta\partial_\theta+u^z\partial_z=u^r\partial_r+u^z\partial_z.$$

We have used in the last equality the fact that for axisymmetric flows the angular component is zero. Hence we get

$$\begin{aligned}(u\cdot\nabla\Omega)\cdot e_r &= u^r\partial_r\Omega\cdot e_r+u^z\partial_z\Omega\cdot e_r\\ &= (u^r\partial_r+u^z\partial_z)(\Omega\cdot e_r)\\ &= u\cdot\nabla\Omega^r,\end{aligned}$$

Where we use $\partial_r e_r=\partial_z e_r=0$. Now it remains to compute $(\Omega\cdot\nabla u)\cdot e_r$. By a straightforward computations we get,

$$\begin{aligned}(\Omega\cdot\nabla u)\cdot e_r &= \Omega^r\,\partial_r u\cdot e_r+\frac{1}{r}\Omega^\theta\,\partial_\theta u\cdot e_r+\Omega^3\,\partial_3 u\cdot e_r\\ &= \Omega^r\partial_r u^r+\Omega^3\partial_3 u^r.\end{aligned}$$

Thus the component $\Omega^r$ obeys to the equation

$$\partial_t\Omega^r+u\cdot\nabla\Omega^r=\Omega^r\partial_r u^r+\Omega^3\partial_3 u^r.$$

From the maximum principle we deduce

$$\|\Omega^r(t)\|_{L^\infty}\leq\int_0^t\big(\|\Omega^r(\tau)\|_{L^\infty}+\|\Omega^3(\tau)\|_{L^\infty}\big)\|\nabla u(\tau)\|_{L^\infty}d\tau.$$

On the other hand the component $\Omega^3$ satisfies the equation

$$\partial_t\Omega^3 + u\cdot\nabla\Omega^3 \quad = \quad \Omega^3\partial_3 u^3 + \Omega^r\partial_r u^3.$$

This leads to

$$\|\Omega^3(t)\|_{L^\infty} \le \int_0^t \big(\|\Omega^3(\tau)\|_{L^\infty} + \|\Omega^r(\tau)\|_{L^\infty}\big)\|\nabla u(\tau)\|_{L^\infty} d\tau.$$

Combining these estimates and using Gronwall's inequality we obtain for every $t\in\mathbb{R}_+$, $\Omega^3(t) = \Omega^r(t) = 0$, which is the desired result.
Under these assumptions the stretching term becomes

$$\begin{aligned}\Omega\cdot\nabla u = \frac{1}{r}\Omega^\theta\partial_\theta(u^r e_r) \quad &= \quad \frac{1}{r}u^r\Omega^\theta e_\theta\\ &= \quad \frac{1}{r}u^r\Omega,\end{aligned}$$

which ends the proof of Proposition 3.2. □

## 4. Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in several steps and it suffices to establish the *a priori* estimates. The hard part of the proof will be the Lipschitz bound of the velocity.

### 4.1. **Some a priori estimates.** We start with the following

**Proposition 4.1.** *Let $u$ be an axisymmetric solution of* (1.1)*, then we have for every $t\in\mathbb{R}_+$,*

i) *Biot-Savart law:*

$$\big\|u^r(t)/r\big\|_{L^\infty} \lesssim \big\|\omega_0/r\big\|_{L^{3,1}}.$$

ii) *Vorticity bound:*

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega_0\|_{L^\infty} e^{Ct\|\omega_0/r\|_{L^{3,1}}}.$$

iii) *Velocity bound:*

$$\|u(t)\|_{L^\infty} \lesssim \big(\|u_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big)e^{\exp Ct\|\omega_0/r\|_{L^{3,1}}}.$$

*Proof.* i) According to Lemma 1 in [15] (see also [7]) one has, for every $x=(x_1,x_2,x_3)\in\mathbb{R}^3$,

$$|u^r(t,x)| \lesssim \int_{|y-x|\le r}\frac{|\omega(t,y)|}{|x-y|^2}dy + r\int_{|y-x|\ge r}\frac{|\omega(t,y)|}{|x-y|^3}dy,$$

with $r=(x_1^2+x_2^2)^{\frac12}$. Thus, if we denote $r'=(y_1^2+y_2^2)^{\frac12}$, one can estimate

$$|u^r(t,x)| \lesssim \int_{|y-x|\le r}\frac{|\omega(t,y)|}{r'}\frac{r'}{|x-y|^2}dy + r\int_{|y-x|\ge r}\frac{|\omega(t,y)|}{r'}\frac{r'-r+r}{|x-y|^3}dy.$$

But since $|r'-r|\le|x-y|$ we have

$$|x-y|\le r \Rightarrow r'\le 2r.$$

This yields in particular

$$|u^r(t,x)|\lesssim r\int_{\mathbb{R}^3}\frac{|\omega(t,y)|}{r'}\frac{1}{|x-y|^2}dy,$$

which can be rewritten as

$$|u^r/r|\lesssim\frac{1}{|\cdot|^2}\star|\omega/r|.$$

As $\frac{1}{|\cdot|^2}\in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, then Young inequalities on $L^{q,p}$ spaces[1] imply

$$\big\|u^r/r\big\|_{L^\infty}\lesssim\big\|\omega/r\big\|_{L^{3,1}}.$$

Since $\omega/r$ satisfies (1.3) then applying Proposition 2.3 gives

$$\big\|u^r/r\big\|_{L^\infty}\lesssim\big\|\omega_0/r\big\|_{L^{3,1}}.$$

ii) From the maximum principle applied to (1.2) one has

$$\|\omega(t)\|_{L^\infty}\le\|\omega_0\|_{L^\infty}+\int_0^t\big\|u^r(\tau)/r\big\|_{L^\infty}\|\omega(\tau)\|_{L^\infty}d\tau.$$

Using Gronwall's lemma and i) gives the desired result.

iii) To estimate $L^\infty$ norm of the velocity we use the argument of Serfati [14],

$$\|u(t)\|_{L^\infty}\le\|\dot S_{-N}u\|_{L^\infty}+\sum_{q\ge -N}\|\dot\Delta_q u\|_{L^\infty},$$

where $N$ is an arbitrary positive integer that will be fixed later. By Bernstein inequality we infer[2]

$$\sum_{q\ge -N}\|\dot\Delta_q u\|_{L^\infty}\lesssim 2^N\|\omega\|_{L^\infty}.$$

On the other hand using the integral equation we get

$$\begin{aligned}\|\dot S_{-N}u\|_{L^\infty}&\le\|\dot S_{-N}u_0\|_{L^\infty}+\int_0^t\|\dot S_{-N}\big(\mathbb{P}(u\cdot\nabla)u\big)\|_{L^\infty}d\tau\\&\lesssim\|u_0\|_{L^\infty}+\sum_{j<-N}\int_0^t\|\dot\Delta_j\big(\mathbb{P}(u\cdot\nabla)u\big)\|_{L^\infty}d\tau,\end{aligned}$$

where $\mathbb{P}$ denotes the Leray's projector over divergence free vector fields. Since $\dot\Delta_j\mathbb{P}$ maps $L^p$ to itself uniformly[3] in $j\in\mathbb{Z}$, we get

$$\|\dot S_{-N}u\|_{L^\infty}\lesssim\|u_0\|_{L^\infty}+2^{-N}\int_0^t\|u(\tau)\|^2_{L^\infty}d\tau.$$

[1]The convolution $L^{p,q}\star L^{p',q'}\longrightarrow L^\infty$ is a bilinear continuous operator, (see [10], page 141 for more details).

[2] We recall the classical fact $\|\dot\Delta_q u\|_{L^p}\approx 2^{-q}\|\dot\Delta_q\omega\|_{L^p}$ uniformly in $q$, for every $p\in[1,+\infty]$.

[3]We stress the fact that this is true for every $p\in[1,+\infty]$ since $\mathbb{P}$ is a Fourier multiplier of degree zero so $\dot\Delta_j\mathbb{P}=\Psi(2^{-j}D)$, where $\Psi\in C_0^\infty$.

Hence we obtain

$$\|u(t)\|_{L^\infty} \lesssim \|u_0\|_{L^\infty} + 2^N \|\omega(t)\|_{L^\infty} + 2^{-N} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau.$$

If we choose $N$ such that

$$2^{2N} \approx 1 + \|\omega(t)\|_{L^\infty}^{-1} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau,$$

then we obtain

$$\|u(t)\|_{L^\infty}^2 \lesssim \|u_0\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau.$$

Thus Gronwall's lemma and the $L^\infty$ bound of the vorticity yield

$$\begin{aligned}\|u(t)\|_{L^\infty} &\lesssim \big(\|u_0\|_{L^\infty} + \|\omega\|_{L_t^\infty L^\infty}\big) e^{Ct\|\omega\|_{L_t^\infty L^\infty}} \\ &\lesssim \big(\|u_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big) e^{\exp Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.\end{aligned}$$

□

4.2. **Lipschitz estimate of the velocity.** The Lipschitz estimate of the velocity is heavily related to the following interpolation result which is the heart of this work:

**Proposition 4.2.** *There exists a decomposition $(\tilde{\omega}_q)_{q\geq -1}$ of the vorticity $\omega$ such that*

i) *For every $t \in \mathbb{R}_+$, $\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x)$.*
ii) *For every $t \in \mathbb{R}_+, \operatorname{div} \tilde{\omega}_q(t,x) = 0$.*
iii) *For every $q \geq -1$ we have $\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q \omega_0\|_{L^\infty} e^{Ct\|\omega_0/r\|_{L^{3,1}}}$.*
iv) *For all $j, q \geq -1$ we have*

$$\|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} \leq C 2^{-|j-q|} e^{CU(t)} \|\Delta_q \omega_0\|_{L^\infty},$$

*with $U(t) := \|u\|_{L_t^1 B_{\infty,1}^1}$ and $C$ an absolute constant.*

*Proof.* We will use for this purpose a new approach similar to [8]. Let $q \geq -1$ and denote by $\tilde{\omega}_q$ the unique global vector-valued solution of the problem

$$\left\{\begin{array}{l}\partial_t \tilde{\omega}_q + (u\cdot\nabla)\tilde{\omega}_q = \tilde{\omega}_q \cdot \nabla u \\ \tilde{\omega}_{q|t=0} = \Delta_q \omega_0.\end{array}\right. \tag{4.1}$$

Since $\operatorname{div} \Delta_q \omega_0 = 0$, then it follows from Proposition 3.2 that $\operatorname{div} \tilde{\omega}_q(t,x) = 0$. On the other hand we have by linearity and uniqueness

$$\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x). \tag{4.2}$$

We will now rewrite the equation (4.1) under a suitable form.

As $\Delta_q\omega_0 = \text{curl}\,\Delta_q u_0$ and $\Delta_q u_0$ is axisymmetric then we obtain from Proposition 3.1 that $(\Delta_q\omega_0)\times e_\theta = (0,0,0)$. This leads in view of Proposition 3.2 to $\tilde\omega_q(t)\times e_\theta = (0,0,0)$ and

$$\begin{cases} \partial_t\tilde\omega_q + (u\cdot\nabla)\tilde\omega_q = \frac{u^r}{r}\tilde\omega_q \\ \tilde\omega_{q|t=0} = \Delta_q\omega_0. \end{cases} \tag{4.3}$$

Applying the maximum principle and using Proposition 4.1 we obtain

$$\begin{aligned} \|\tilde\omega_q(t)\|_{L^\infty} &\le \|\Delta_q\omega_0\|_{L^\infty} e^{\int_0^t \|u^r(\tau)/r\|_{L^\infty} d\tau} \\ &\le \|\Delta_q\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}. \end{aligned} \tag{4.4}$$

This concludes the proof of i-iii) of the proposition.

Let us now move to the proof of iv) which is the main property of the decomposition above. Remark first that the desired estimate is equivalent to

$$\|\Delta_j\tilde\omega_q(t)\|_{L^\infty} \le C2^{j-q}e^{CU(t)}\|\Delta_q\omega_0\|_{L^\infty} \tag{4.5}$$

and

$$\|\Delta_j\tilde\omega_q(t)\|_{L^\infty} \le C2^{q-j}e^{CU(t)}\|\Delta_q\omega_0\|_{L^\infty}. \tag{4.6}$$

• *Proof of* (4.5). Applying Proposition A.2 of the appendix to (4.1)

$$\begin{aligned} e^{-CU(t)}\|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\lesssim \|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}} + \\ &\quad + \int_0^t e^{-CU(\tau)}\|\tilde\omega_q\cdot\nabla u(\tau)\|_{B^{-1}_{\infty,\infty}} d\tau. \end{aligned} \tag{4.7}$$

To estimate the integral term we write in view of Bony's decomposition

$$\begin{aligned} \|\tilde\omega_q\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} &\le \|T_{\tilde\omega_q}\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} + \|T_{\nabla u}\cdot\tilde\omega_q\|_{B^{-1}_{\infty,\infty}} \\ &+ \|\mathcal{R}(\tilde\omega_q\cdot\nabla, u)\|_{B^{-1}_{\infty,\infty}} \\ &\lesssim \|\nabla u\|_{L^\infty}\|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}} + \|\mathcal{R}(\tilde\omega_q\cdot\nabla, u)\|_{B^{-1}_{\infty,\infty}}. \end{aligned}$$

Since $\text{div}\,\tilde\omega_q = 0$, then the remainder term can be treated as follows

$$\begin{aligned} \|\mathcal{R}(\tilde\omega_q\cdot\nabla, u)\|_{B^{-1}_{\infty,\infty}} &= \|\text{div}\,\mathcal{R}(\tilde\omega_q\otimes, u)\|_{B^{-1}_{\infty,\infty}} \\ &\lesssim \sup_k \sum_{j\ge k-3}\|\Delta_j\tilde\omega_q\|_{L^\infty}\|\widetilde\Delta_j u\|_{L^\infty} \\ &\lesssim \|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}}\|u\|_{B^1_{\infty,1}}. \end{aligned}$$

Il follows that

$$\|\tilde\omega_q\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} \lesssim \|u\|_{B^1_{\infty,1}}\|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}}$$

Inserting this estimate into (4.7) we get

$$\begin{aligned} e^{-CU(t)}\|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\lesssim \|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}}\\ &+ \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} e^{-CU(\tau)}\|\tilde\omega_q(\tau)\|_{B^{-1}_{\infty,\infty}}d\tau. \end{aligned}$$

Hence we obtain by Gronwall's inequality

$$\begin{aligned} \|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\leq C\|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}} e^{CU(t)}\\ &\leq C2^{-q}\|\Delta_q\omega_0\|_{L^\infty} e^{CU(t)}. \end{aligned}$$

This gives by definition

$$\|\Delta_j\tilde\omega_q(t)\|_{L^\infty} \leq C2^{j-q}\|\Delta_q\omega_0\|_{L^\infty} e^{CU(t)}.$$

• *Proof of* (4.6). Since $w_q(t)\times e_\theta = (0,0,0)$ the solution $\tilde\omega_q$ has two components in the cartesian basis, $\tilde\omega_q = (\tilde\omega_q^1, \tilde\omega_q^2, 0)$. The analysis will be exactly the same for both components, so we will deal only with the first one.

From the identity $\frac{u^r}{r} = \frac{u^1}{x_1} = \frac{u^2}{x_2}$, which is an easy consequence of $u^\theta = 0$, it is plain that the functions $\tilde\omega_q^1$ is solution of

$$\begin{cases} \partial_t\tilde\omega_q^1 + (u\cdot\nabla)\tilde\omega_q^1 = u^2\frac{\tilde\omega_q^1}{x_2},\\ \tilde\omega^1_{q|t=0} = \Delta_q\omega_0^1. \end{cases}$$

Unfortunately, we are not able to close the estimate in Besov space $B^1_{\infty,\infty}$ due to the invalidity of a commutator estimate in Proposition A.2 for the limiting case $s=1$. Nevertherless we will be able to do it for Besov space $B^1_{\infty,1}$.

In view of Proposition A.2 in the appendix we have

$$e^{-CU(t)}\|\tilde\omega_q^1(t)\|_{B^1_{\infty,1}} \lesssim \|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}} + \int_0^t e^{-CU(\tau)}\Big\|u^2\frac{\tilde\omega_q^1}{x_2}(\tau)\Big\|_{B^1_{\infty,1}}d\tau. \tag{4.8}$$

To estimate the integral term we write from Bony's decomposition,

$$\Big\|u^2\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}} \leq \Big\|T_{\frac{\tilde\omega_q^1}{x_2}}u^2\Big\|_{B^1_{\infty,1}} + \Big\|T_{u^2}\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}} + \Big\|\mathcal{R}(u^2,\tilde\omega_q^1/x_2)\Big\|_{B^1_{\infty,1}}.$$

To estimate the first paraproduct we write by definition,

$$\begin{aligned} \Big\|T_{\frac{\tilde\omega_q^1}{x_2}}u^2\Big\|_{B^1_{\infty,1}} &\lesssim \sum_j 2^j\|S_{j-1}(\tilde\omega_q^1/x_2)\|_{L^\infty}\|\Delta_j u^2\|_{L^\infty}\\ &\lesssim \|u\|_{B^1_{\infty,1}}\|\tilde\omega_q^1/x_2\|_{L^\infty}. \end{aligned} \tag{4.9}$$

The remainder term is estimated as follows,

$$\begin{aligned}\|\mathcal{R}(u^2,\tilde{\omega}_q^1/x_2)\|_{B^1_{\infty,1}} &\lesssim \sum_{k\geq j-3} 2^j\|\Delta_k u^2\|_{L^\infty}\|\widetilde{\Delta}_k(\tilde{\omega}_q^1/x_2)\|_{L^\infty}\\ &\lesssim \|u\|_{B^1_{\infty,1}}\|\tilde{\omega}_q^1/x_2\|_{L^\infty}.\end{aligned}\tag{4.10}$$

The treatment of the second term is more subtle and needs the axisymmetry of the vector field $u$. By definition we have

$$\Big\|T_{u^2}\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\lesssim\sum_{j\in\mathbb{N}}2^j\|S_{j-1}u^2(x)\Delta_j(\tilde{\omega}_q^1(x)/x_2)\|_{L^\infty}.$$

Now we write

$$\begin{aligned}S_{j-1}u^2(x)\Delta_j(\tilde{\omega}_q^1(x)/x_2) &= S_{j-1}u^2(x)\Delta_j\tilde{\omega}_q^1(x)/x_2+S_{j-1}u^2(x)\Big[\Delta_j,\frac{1}{x_2}\Big]\tilde{\omega}_q^1\\ &:= \mathrm{I}_j(x)+\mathrm{II}_j(x).\end{aligned}$$

Since $S_{j-1}u$ est axisymmetric then it follows from Proposition 3.1 that $S_{j-1}u^2(x_1,0,z)=0$. Thus from Taylor formula we get

$$\|\mathrm{I}_j\|_{L^\infty}\lesssim\|\nabla u\|_{L^\infty}\|\Delta_j\tilde{\omega}_q^1\|_{L^\infty}.$$

This yields

$$\sum_j 2^j\|\mathrm{I}_j\|_{L^\infty}\lesssim\|\nabla u\|_{L^\infty}\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}.\tag{4.11}$$

For the commutator term $\mathrm{II}_j$ we write by definition

$$\begin{aligned}\mathrm{II}_j(x) &= S_{j-1}u^2(x)/x_2\;2^{3j}\int_{\mathbb{R}^3}h(2^j(x-y))(x_2-y_2)\tilde{\omega}_q^1(y)/y_2dy\\ &= 2^{-j}(S_{j-1}u^2(x)/x_2)\,2^{3j}\tilde{h}(2^j\cdot)\star(\tilde{\omega}_q^1/y_2)(x),\end{aligned}$$

with $\tilde{h}(x)=x_2h(x)$. Now we claim that for every $f\in\mathcal{S}'$ we have

$$2^{3j}\tilde{h}(2^j\cdot)\star f=\sum_{|j-k|\leq1}2^{3j}\tilde{h}(2^j\cdot)\star\Delta_k f.$$

Indeed, we have $\widehat{\tilde{h}}(\xi)=i\partial_{\xi_2}\widehat{h}(\xi)=i\partial_{\xi_2}\varphi(\xi)$. It follows that $supp\,\widehat{\tilde{h}}\subset supp\,\varphi$. So we get $2^{3j}\tilde{h}(2^j\cdot)\star\Delta_k f=0$, for $|j-k|\geq2$. This leads to

$$\begin{aligned}\sum_{j\in\mathbb{N}}2^j\|\mathrm{II}_j\|_{L^\infty} &\lesssim \sum_{|j-k|\leq1}\|S_{j-1}u^2/x_2\|_{L^\infty}\|\Delta_k(\tilde{\omega}_q^1/x_2)\|_{L^\infty}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\tilde{\omega}_q^1/x_2\|_{B^0_{\infty,1}}.\end{aligned}\tag{4.12}$$

Using (4.11) et (4.12) one obtains

$$\big\|T_{u^2}\tilde{\omega}_q^1/x_2\big\|_{B^1_{\infty,1}}\lesssim\|\nabla u\|_{L^\infty}\big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}+\|\tilde{\omega}_q^1/x_2\|_{B^0_{\infty,1}}\big).\tag{4.13}$$

Putting together (4.9) (4.10) and (4.13) we find

$$\Big\|u^2\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\lesssim\|u\|_{B^1_{\infty,1}}\big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}+\|\tilde{\omega}_q^1/x_2\|_{B^0_{\infty,1}}\big)$$

Therefore we get from (4.8),

$$\begin{aligned}e^{-CU(t)}\|\tilde{\omega}_q^1(t)\|_{B^1_{\infty,1}} &\lesssim \|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}}+\int_0^t e^{-CU(\tau)}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}\|u(\tau)\|_{B^1_{\infty,1}}d\tau\\ &+ \int_0^t e^{-CU(\tau)}\|u(\tau)\|_{B^1_{\infty,1}}\|\tilde{\omega}_q^1(\tau)/x_2\|_{B^0_{\infty,1}}d\tau.\end{aligned}$$

According to Gronwall's inequality we have

$$\|\tilde{\omega}_q^1(t)\|_{B^1_{\infty,1}}\lesssim e^{CU(t)}\Big(\|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}}+\|\tilde{\omega}_q^1/x_2\|_{L^\infty_t B^0_{\infty,1}}\Big). \tag{4.14}$$

Let us now estimate $\|\tilde{\omega}_q^1/x_2\|_{L^\infty_t B^0_{\infty,1}}$. It is easy to check that $\tilde{\omega}_q^1/x_2$ is advected by the flow, that is

$$\begin{cases}(\partial_t+u\cdot\nabla)\frac{\tilde{\omega}_q^1}{x_2}=0\\ \frac{\tilde{\omega}_q^1}{x_2}|_{t=0}=\frac{\Delta_q\omega_0^1}{x_2}.\end{cases}$$

Thus we deduce from Proposition A.2,

$$\big\|\tilde{\omega}_q^1(t)/x_2\big\|_{B^0_{\infty,1}}\leq\big\|\Delta_q\omega_0^1/x_2\big\|_{B^0_{\infty,1}}e^{CU(t)}. \tag{4.15}$$

At this stage we need the following lemma.

**Lemma 4.3.** *Under the assumptions on $u_0$, one has*

$$\Big\|\Delta_q\omega_0^1/x_2\Big\|_{B^0_{\infty,1}}\lesssim 2^q\|\Delta_q\omega_0\|_{L^\infty}. \tag{4.16}$$

*Proof.* Since $u_0$ is axisymmetric then according to Proposition 3.1, $\Delta_q u_0$ is too. Consequently $\Delta_q\omega_0$ is the curl of an axisymmetric vector field and then by Proposition 3.1 and Taylor expansion

$$\Delta_q\omega_0^1(x_1,x_2,z)=x_2\int_0^1(\partial_{x_2}\Delta_q\omega_0^1)(x_1,\tau x_2,z)d\tau.$$

Hence we get in view of Proposition A.1

$$\begin{aligned}\big\|\Delta_q\omega_0^1/x_2\big\|_{B^0_{\infty,1}} &\leq \int_0^1\|(\partial_{x_2}\Delta_q\omega_0^1)(\cdot,\tau\cdot,\cdot)\|_{B^0_{\infty,1}}d\tau\\ &\lesssim \|\partial_{x_2}\Delta_q\omega_0^1\|_{B^0_{\infty,1}}\int_0^1(1-\log\tau)d\tau\\ &\lesssim 2^q\|\Delta_q\omega_0^1\|_{L^\infty},\end{aligned}$$

as claimed. □

Coming back to the proof of (4.6). We put together (4.14), (4.15) and (4.16) to get

$$\|\tilde{\omega}_q^1(t)\|_{B^1_{\infty,1}} \leq C2^q \|\Delta_q \omega_0\|_{L^\infty} e^{CU(t)}.$$

This can be written as

$$\|\Delta_j \tilde{\omega}_q^1(t)\|_{L^\infty} \leq C2^{q-j} e^{CU(t)} \|\Delta_q \omega_0\|_{L^\infty}, \tag{4.17}$$

which is (4.6). □

In the next proposition we give some precise estimates of the velocity.

**Proposition 4.4.** *The Euler solution with initial data $u_0 \in B_{p,1}^{1+\frac{3}{p}}$ such that $\frac{\omega_0}{r} \in L^{3,1}$ satisfies for every $t \in \mathbb{R}_+$,*

i) *Case $p = \infty$,*

$$\|\omega(t)\|_{B^0_{\infty,1}} + \|u(t)\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t}.$$

ii) *Case $1 \leq p < \infty$,*

$$\|u(t)\|_{B_{p,1}^{1+\frac{3}{p}}} \leq C_0 e^{e^{\exp C_0 t}},$$

*with $C_0$ depends on the norms of $u_0$.*

*Proof.* i) Let $N$ be a fixed positive integer that will be carefully chosen later. Then we have from (4.2)

$$\begin{aligned} \|\omega(t)\|_{B^0_{\infty,1}} &\leq \sum_j \|\Delta_j \sum_q \tilde{\omega}_q(t)\|_{L^\infty} \\ &\leq \sum_{|j-q|\geq N} \|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} + \sum_{|j-q|<N} \|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{4.18}$$

To estimate the first term we use Proposition 4.2 and the convolution inequality for the series

$$\mathrm{I} \lesssim 2^{-N} \|\omega_0\|_{B^0_{\infty,1}} e^{CU(t)}. \tag{4.19}$$

To estimate the term II we use two facts: the first one is that the operator $\Delta_j$ maps uniformly $L^\infty$ into itself while the second is the $L^\infty$ estimate (4.4),

$$\begin{aligned} \mathrm{II} &\lesssim \sum_{|j-q|<N} \|\tilde{\omega}_q(t)\|_{L^\infty} \\ &\lesssim e^{C_0 t} \sum_{|j-q|<N} \|\Delta_q \omega_0\|_{L^\infty} \\ &\lesssim e^{C_0 t} N \|\omega_0\|_{B^0_{\infty,1}}. \end{aligned} \tag{4.20}$$

Combining this estimate with (4.20), (4.19) and (4.18) we obtain

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim 2^{-N} e^{CU(t)} + N e^{C_0 t}.$$

Putting

$$N = \Big[CU(t)\Big] + 1,$$

we obtain

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \big(U(t)+1\big)e^{C_0 t}.$$

On the other hand we have

$$\|u\|_{B^1_{\infty,1}} \lesssim \|u\|_{L^\infty} + \|\omega\|_{B^0_{\infty,1}},$$

which yields in view of Proposition 4.1,

$$\begin{aligned}\|u(t)\|_{B^1_{\infty,1}} &\lesssim \|u(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}\\ &\le C_0 e^{\exp C_0 t} + C_0 e^{C_0 t}\int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau.\end{aligned}$$

Hence we obtain by Gronwall's inequality

$$\|u(t)\|_{B^1_{\infty,1}} \le C_0 e^{\exp C_0 t},$$

which gives in turn

$$\|\omega(t)\|_{B^0_{\infty,1}} \le C_0 e^{\exp C_0 t}.$$

This concludes the first part of Proposition 4.4.

ii) Applying Proposition A.2 to the vorticity equation we get

$$e^{-CU_1(t)}\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CU_1(\tau)}\|\omega\cdot\nabla u(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau. \tag{4.21}$$

As $\omega = \text{curl } u$, we have

$$\|\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega\|_{B^{\frac{3}{p}}_{p,1}}\|\nabla u\|_{L^\infty}. \tag{4.22}$$

Indeed, from Bony's decomposition we write

$$\begin{aligned}\|\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} &\le \|T_{\nabla u}\cdot\omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} + \|\mathcal{R}(\omega,\nabla u)\|_{B^{\frac{3}{p}}_{p,1}}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

From the definition we write

$$\begin{aligned}\|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} &\lesssim \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|S_{q-1}\omega\|_{L^\infty}\|\nabla\Delta_q u\|_{L^p}\\ &\lesssim \|\omega\|_{L^\infty}\sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|\Delta_q\omega\|_{L^p}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

We have used here the fact that for $p \in [1,\infty]$ and $q \in \mathbb{N}$ the composition operator $\Delta_q R : L^p \to L^p$ is continuous uniformly with respect to $p$ and $q$, where $R$ denotes Riesz transform. Combining (4.21) and (4.22) we find,

$$e^{-CU_1(t)}\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CU_1(\tau)}\|\omega(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\|\nabla u(\tau)\|_{L^\infty}d\tau.$$

Gronwall's inequality yields

$$\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \leq \|u_0\|_{B^{\frac{3}{p}+1}_{p,1}} e^{C\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau} \leq C_0 e^{e^{\exp C_0 t}}.$$

Let us estimate the velocity. We write

$$\begin{aligned}\|u(t)\|_{B^{1+\frac{3}{p}}_{p,1}} &\lesssim \|\Delta_{-1}u\|_{L^p} + \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}2^q\|\Delta_q u\|_{L^p}\\ &\lesssim \|u(t)\|_{L^p} + \|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

Thus it remains to estimate $\|u\|_{L^p}$. For $1 < p < \infty$, since Riesz transforms act continuously on $L^p$, we get

$$\begin{aligned}\|u(t)\|_{L^p} &\leq \|u_0\|_{L^p} + C\int_0^t \|u\cdot\nabla u(\tau)\|_{L^p}d\tau\\ &\lesssim \|u_0\|_{L^p} + \int_0^t \|u(\tau)\|_{L^p}\|\nabla u(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

It suffices now to use Gronwall's inequality.
For the case $p = 1$, we write

$$\begin{aligned}\|u(t)\|_{L^1} &\leq \|\dot{S}_0 u(t)\|_{L^1} + \sum_{q\geq 0}\|\dot{\Delta}_q u(t)\|_{L^1}\\ &\lesssim \|\dot{S}_0 u(t)\|_{L^1} + \sum_{q\geq 0} 2^{-q}\|\dot{\Delta}_q \nabla u(t)\|_{L^1}\\ &\lesssim \|\dot{S}_0 u(t)\|_{L^1} + \|\omega(t)\|_{L^1}.\end{aligned}$$

However, it is easy to see that

$$\|\omega(t)\|_{L^1} \leq \|\omega_0\|_{L^1} e^{\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau}.$$

Concerning $\dot{S}_0 u$ we use the equation on $u$ leading to

$$\begin{aligned}\|\dot{S}_0 u(t)\|_{L^1} &\lesssim \|\dot{S}_0 u_0\|_{L^1} + \sum_{q\leq -1}\|\dot{\Delta}_q\mathbb{P}((u\cdot\nabla)u(t))\|_{L^1}\\ &\lesssim \|u_0\|_{L^1} + \sum_{q\leq -1} 2^q\|\dot{\Delta}_q(u\otimes u(t))\|_{L^1}\\ &\lesssim \|u_0\|_{L^1} + \|u(t)\|_{L^2}^2\\ &\lesssim \|u_0\|_{L^1} + \|u_0\|_{L^2}^2.\end{aligned}$$

This yields

$$\|u(t)\|_{L^1} \leq C_0 e^{e^{\exp C_0 t}}.$$

The proof is now achieved.

□

## Appendix A. Appendix

The following result describes the anisotropic dilatation in Besov spaces.

**Proposition A.1.** *Let $f:\mathbb{R}^3\to\mathbb{R}$ be a function belonging to $B^0_{\infty,1}$ and denote by $f_\lambda(x_1,x_2,x_3)=f(\lambda x_1,x_2,x_3)$. Then, there exists an absolute constant $C>0$ such that for all $\lambda\in]0,1[$*

$$\|f_\lambda\|_{B^0_{\infty,1}}\le C(1-\log\lambda)\|f\|_{B^0_{\infty,1}}.$$

*Proof.* Let $q\ge -1$, we denote by $f_{q,\lambda}=(\Delta_q f)_\lambda$. From the definition we have

$$\begin{aligned}\|f_\lambda\|_{B^0_{\infty,1}} &= \|\Delta_{-1}f_\lambda\|_{L^\infty}+\sum_{j\in\mathbb{N}}\|\Delta_j f_\lambda\|_{L^\infty}\\ &\le C\|f\|_{L^\infty}+\sum_{\substack{j\in\mathbb{N}\\ q\ge -1}}\|\Delta_j f_{q,\lambda}\|_{L^\infty}.\end{aligned}$$

For $j,q\in\mathbb{N}$, the Fourier transform of $\Delta_j f_{q,\lambda}$ is supported in the set

$$\Big\{|\xi_1|+|\xi'|\approx 2^j\quad\text{and}\quad \lambda^{-1}|\xi_1|+|\xi'|\approx 2^q\Big\},$$

where $\xi'=(\xi_2,\xi_3)$. A direct consideration shows that this set is empty if $2^q\lesssim 2^j$ or $2^{j-q}\lesssim\lambda$. For $q=-1$ the set is empty if $j\ge n_0$, this last number is absolute. Thus we get for an integer $n_1$

$$\begin{aligned}\|f_\lambda\|_{B^0_{\infty,1}} &\lesssim \|f\|_{L^\infty}+\sum_{\substack{q-n_1+\log\lambda\le j\\ j\le q+n_1}}\|\Delta_j f_{q,\lambda}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty}+(n_1-\log\lambda)\sum_q\|f_{q,\lambda}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty}+(n_1-\log\lambda)\sum_q\|f_q\|_{L^\infty}\\ &\lesssim (1-\log\lambda)\|f\|_{B^0_{\infty,1}}.\end{aligned}$$

□

The following proposition describes the propagation of Besov regularity for transport equation.

**Proposition A.2.** *Let $s\in]-1,1[, p,r\in[1,\infty]$ and $u$ be a smooth divergence free vector field. Let $f$ be a smooth solution of the transport equation*

$$\partial_t f+u\cdot\nabla f=g,\; f_{|t=0}=f_0,$$

*such that $f_0\in B^s_{p,r}(\mathbb{R}^3)$ and $g\in L^1_{\rm loc}(\mathbb{R}_+;B^s_{p,r})$. Then $\forall t\in\mathbb{R}_+$,*

$$\|f(t)\|_{B^s_{p,r}}\le Ce^{CU_1(t)}\Big(\|f_0\|_{B^s_{p,r}}+\int_0^t e^{-CU_1(\tau)}\|g(\tau)\|_{B^s_{p,r}}d\tau\Big), \tag{A.1}$$

*where* $U_1(t)=\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau$ *and* $C$ *is a constant depending on* $s$.
*The above estimate holds also true in the limiting cases:*

$$s=-1,r=\infty,p\in[1,\infty])\quad or\quad s=1,r=1,p\in[1,\infty],$$

*provided that we change* $U_1$ *by* $U(t):=\|u\|_{L^1_tB^1_{\infty,1}}$.
*In addition if* $f=\operatorname{curl}u$, *then the above estimate (A.1) holds true for all* $s\in[1,+\infty[$.

*Proof.* We will only restrict ourselves to the proof of the limiting cases $s=\mp1$. The remainder cases are done for example in [5, 17].
We start with localizing in frequency the equation leading to,

$$\begin{aligned}\partial_t\Delta_qf+(u\cdot\nabla)\Delta_qf&=\Delta_qg+(u\cdot\nabla)\Delta_qf-\Delta_q\big(u\cdot\nabla f\big)\\&=\Delta_qg-[\Delta_q,u\cdot\nabla]f.\end{aligned}$$

Taking the $L^p$ norm, then the zero divergence of the flow gives

$$\|\Delta_qf(t)\|_{L^p}\le\|\Delta_qf_0\|_{L^p}+\int_0^t\|\Delta_qg\|_{L^p}d\tau+\int_0^t\Big\|[\Delta_q,u\cdot\nabla]f\Big\|_{L^p}d\tau.$$

From Bony's decomposition, the commutator may be decomposed as follows

$$\begin{aligned}[]
[\Delta_q,u\cdot\nabla]f&=\Delta_q\mathcal{R}(u^j,\partial_jf)+\Delta_qT_{\partial_jf}u^j-T'_{\Delta_q\partial_jf}u^j+[\Delta_q,T_{u^j}]\partial_jf\\
&:=\sum_{i=1}^4\mathcal{R}_q^i,
\end{aligned}$$

where $T'_uv$ stands for $T_uv+\mathcal{R}(u,v)$.To treat the first term $\mathcal{R}_q^1$, we write from the definition

$$\mathcal{R}_q^1=\sum_{k\ge q-3}\Delta_q\partial_j(\Delta_kf\widetilde{\Delta}_ku^j).$$

According to Bernstein inequalities we get for $s=-1$,

$$\sup_{q\ge-1}2^{-q}\|\mathcal{R}_q^1\|_{L^p}\lesssim\|f\|_{B^{-1}_{p,\infty}}\|u\|_{B^1_{\infty,1}}.\tag{A.2}$$

To estimate $\mathcal{R}_q^2$, we write by definition

$$\mathcal{R}_q^2=\Delta_qT_{\partial_jf}v^j=\sum_{|q-k|\le4}\Delta_q(S_{k-1}\partial_jf\Delta_ku^j).$$

Applying Bernstein and Young inequalities leads to

$$\begin{aligned}\sup_q2^{-q}\|\mathcal{R}_q^2\|_{L^p}&\lesssim\sup_q2^{-q}\|S_{q-1}f\|_{L^p}2^q\|\Delta_qu^j\|_{L^\infty}\\&\lesssim\|u\|_{B^1_{\infty,\infty}}\sup_q\sum_{-1\le m\le q-2}2^{m-q}2^{-m}\|\Delta_mf\|_{L^p}\\&\lesssim\|f\|_{B^{-1}_{p,\infty}}\|u\|_{B^1_{\infty,\infty}}.\end{aligned}\tag{A.3}$$

It is easy to verify from the definition that $\mathcal{R}_q^3$ can be rewritten like

$$\mathcal{R}_q^3=T'_{\Delta_q\partial_jf}u^j=\sum_{k\ge q-2}S_{k+2}\Delta_q\partial_jf\Delta_ku^j.$$

Thus applying Bernstein inequality one has

$$2^{-q}\|\mathcal{R}_q^3\|_{L^\infty} \lesssim 2^{-q}\|\Delta_q f\|_{L^p} \sum_{k\geq q-2} 2^{q-k}\, 2^k \|\Delta_k u\|_{L^\infty},$$

Therefore we get from the convolution inequality

$$\sup_{q\geq -1} 2^{-q}\|\mathcal{R}_q^3\|_{L^p} \lesssim \|f\|_{B^{-1}_{p,\infty}} \|u\|_{B^1_{\infty,\infty}}. \tag{A.4}$$

For the last term we write

$$\mathcal{R}_q^4 = [\Delta_q, T_{u^j}]\partial_j f = \sum_{|k-q|\leq 4} [\Delta_q, S_{k-1}u^j]\Delta_k \partial_j f.$$

The following is classical (see for example [5]),

$$\begin{aligned} \|[S_{k-1}u^j, \Delta_q]\Delta_k\partial_j f\|_{L^\infty} &\lesssim 2^{-q}\|\nabla S_{k-1}u\|_{L^\infty}\|\partial_j \Delta_k f\|_{L^p} \\ &\lesssim 2^{k-q}\|\nabla u\|_{L^\infty}\|\Delta_k f\|_{L^p}. \end{aligned}$$

This yields

$$\sup_{q\geq -1} 2^{-q}\|\mathcal{R}_q^4\|_{L^\infty} \lesssim \|f\|_{B^s_{p,\infty}}\|\nabla u\|_{L^\infty}. \tag{A.5}$$

Putting together the estimates (A.2), (A.3), (A.4) and (A.5) gives

$$\sup_{q\geq -1} 2^{-q}\Big\|[\Delta_q, u\cdot\nabla]f\Big\|_{L^p} \lesssim \|f\|_{B^{-1}_{p,\infty}}\|u\|_{B^1_{\infty,1}}.$$

This implies

$$\|f(t)\|_{B^{-1}_{p,\infty}} \lesssim \|f_0\|_{B^s_{\infty,\infty}} + \int_0^t \|g(\tau)\|_{B^{-1}_{p,\infty}} d\tau + \int_0^t \|f(\tau)\|_{B^{-1}_{p,\infty}}\|u(\tau)\|_{B^1_{\infty,1}} d\tau.$$

It suffices now to use Gronwall's inequality in order to the desired the result. Let us now move to the case $s = 1$ that will be briefly explained. We estimate $\mathcal{R}_q^1$ as follows

$$\begin{aligned} \sum_q 2^q\|\mathcal{R}_q^1\|_{L^p} &\lesssim \sum_{k\geq q-3} 2^{q-k}2^k\|\Delta_k f\|_{L^p}2^k\|\widetilde{\Delta}_k u_j\|_{L^\infty} \\ &\lesssim \|f\|_{B^1_{p,1}}\|u\|_{B^1_{\infty,\infty}}. \end{aligned}$$

Concerning the second term we write

$$\begin{aligned} \sum_q 2^q\|\mathcal{R}_q^2\|_{L^p} &\lesssim \sum_q 2^q\|S_{q-1}\partial_j f\|_{L^p}\|\Delta_q u^j\|_{L^\infty} \\ &\lesssim \|\nabla f\|_{L^p}\|u\|_{B^1_{\infty,1}} \\ &\lesssim \|f\|_{B^1_{p,1}}\|u\|_{B^1_{\infty,1}}. \end{aligned}$$

The third and the last terms are treated similarly to the first case. □

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